INF721

2024/2

UFV

Deep Learning

L12: Normalization

Logistics

Announcements

- PA3 is due on Oct 30th, Wednesday, 11:59pm

Last Lecture

- Pooling Layers
  - Max Pooling and Average Pooling
- Classic CNNs
  - LeNet-5, AlexNet and VGG-16
- Residual Neural Networks

Lecture Outline

- Normalization
  - Input Normalization
  - Batch Normalization
  - Layer Normalization
- Recurrent Neural Networks

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Image Normalization

You might have noticed that in the first two programming assigments we've normalized the inputs images by dividing all pixel values by 255:

206	205	247	245	244
244	161	137	244	254
192	154	75	200	249
90	109	96	143	223
67	69	107	196	236

Original image

image / 255 →

0.80	0.80	0.96	245	0.96
0.95	0.63	0.53	0.95	0.99
0.75	0.60	0.29	0.78	0.97
0.35	0.42	0.37	0.56	0.87
0.26	0.27	0.41	0.76	0.92

Normalized image

This type of normalization makes the learning process faster, because we are bringing the input values close to zero!

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Input Normalization

Often we encounter datasets in which different input variables span very different ranges:

House Price Prediction Dataset

Size (m2)	Number of Beds.	Nearest Subway Station (m)	Price (1000's of USD)
152	4	7200	1550
229	3	3000	2286
84	1	1500	2930
95	3	12000	196
...	...		...

Such variations can make gradient descent training much more challenging!

- Assume Linear Regression with SGD :

$$\mathbf{w} = \mathbf{w} - \alpha \frac{\partial L}{\partial \mathbf{w}}$$

$$w = [0,0,0]$$
$$\alpha = 0.1$$

$$\mathbf{w} = [0,0,0] - 0.1(\hat{y}^{(i)} - y^{(i)})\mathbf{x}^{(i)}$$

$$\mathbf{w} = [0,0,0] - 0.1(0 - 1550)\mathbf{x}^{(i)}$$

$$\mathbf{w} = [0,0,0] + 155 \cdot [152,4,7200]$$

Changes in w_3 afect much more the output than w_1 and w_2

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Input Normalization

When the input data X is **not** normalized, the error surface will have very different curvatures along different axis:

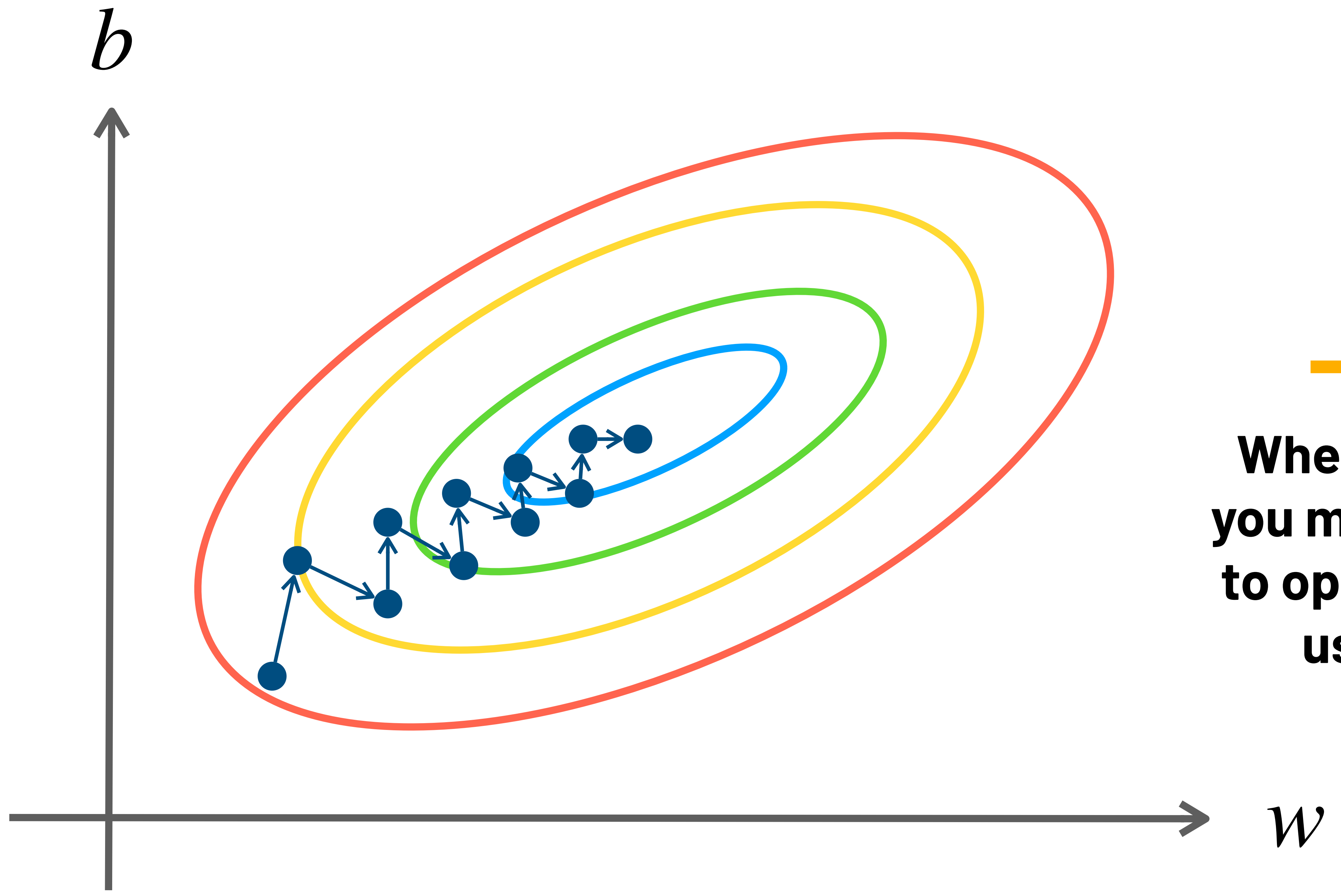


Figure 1: The curvature of the w axis is much larger than the curvature of the b axis.

Normalize X

When you normalize your data, you make the loss surface easier to optimize, so you can typically use larger learning rates.

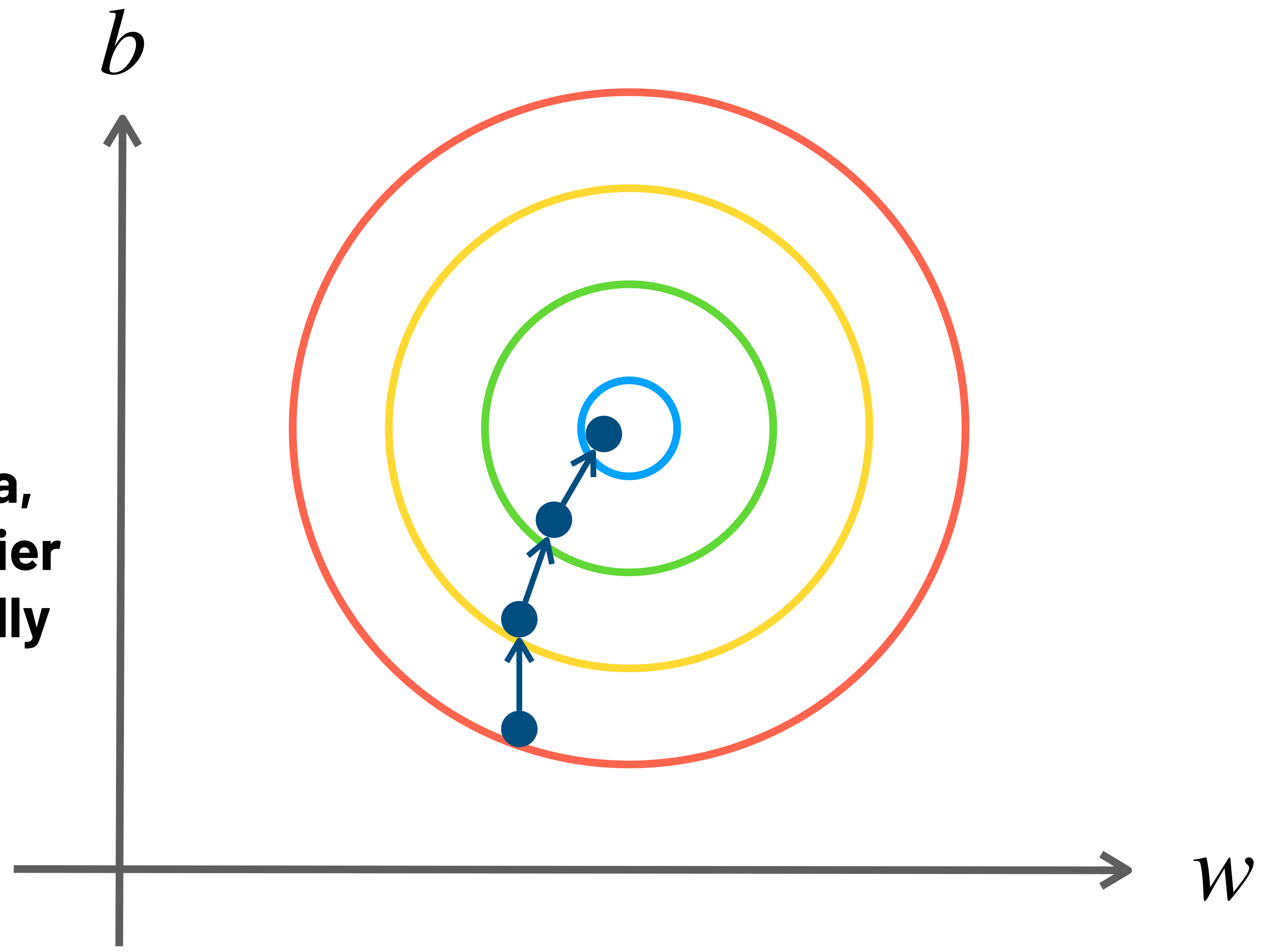
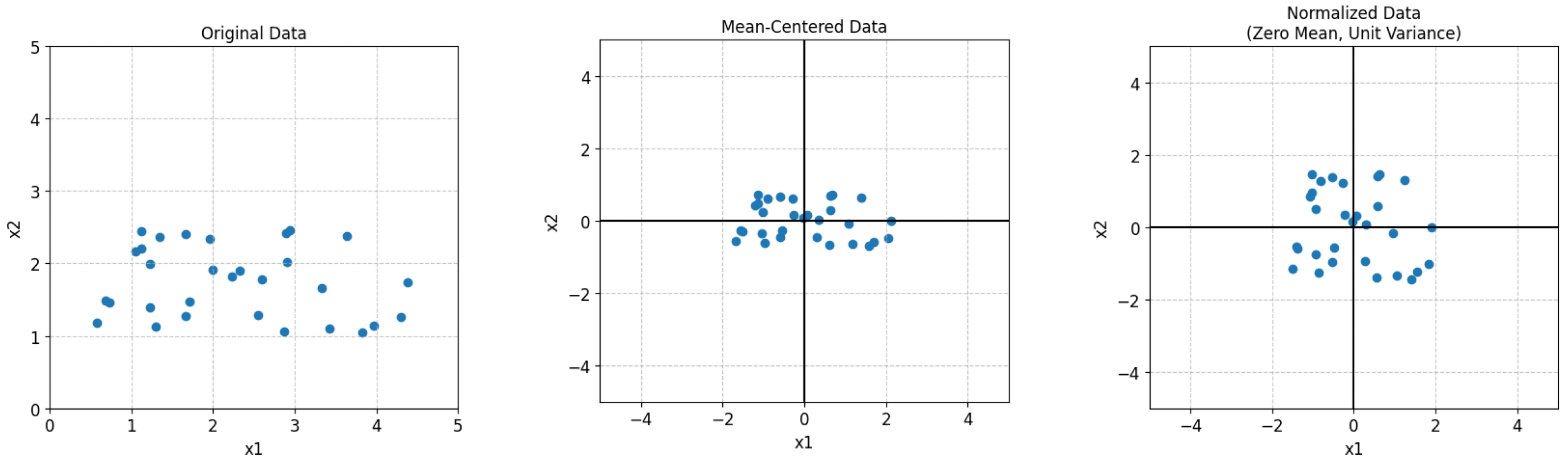


Figure 2: The curvature of the w axis is equal to the curvature of the b axis.

How to Normalize the Input Data

To normalize your data, you need to make your examples have mean $\mu = 0$ and std dev. $\sigma = 1$:



Note that the same values of μ and σ must be used to normalize the training, validation and test sets!

1. Subtract the mean:

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i)} - \mu$$

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i)} - (\frac{1}{m}\sum_{i=1}^{m}\mathbf{x}^{(i)})$$

2. Divide std. deviation:

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i)}/\sigma$$

$$\mathbf{x}^{(i)} = \frac{\mathbf{x}^{(i)}}{\sqrt{(\frac{1}{m}\sum_{i=1}^{m}((\mathbf{x}^{(i)} - \mu)^2)}}$$

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Example 1: Normalizing Structured Datasets

We can also apply this idea to normalize images, which can be done across channels or not:

House Price Prediction Dataset

Size (m2)	Number of Beds.	Nearest Subway Station (m)
0.2086	1.1470	0.3123
1.5477	0.2294	-0.7164
-0.9738	-1.6059	-1.0838
-0.7825	0.2294	1.4880
...	...	

1. Subtract the mean:

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i)} - \mu$$

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i)} - (\frac{1}{m}\sum_{i=1}^{m}\mathbf{x}^{(i)})$$

2. Divide std. deviation:

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i)}/\sigma$$

$$\mathbf{x}^{(i)} = \frac{\mathbf{x}^{(i)}}{\sqrt{(\frac{1}{m}\sum_{i=1}^{m}((\mathbf{x}^{(i)} - \mu)^2)}}$$

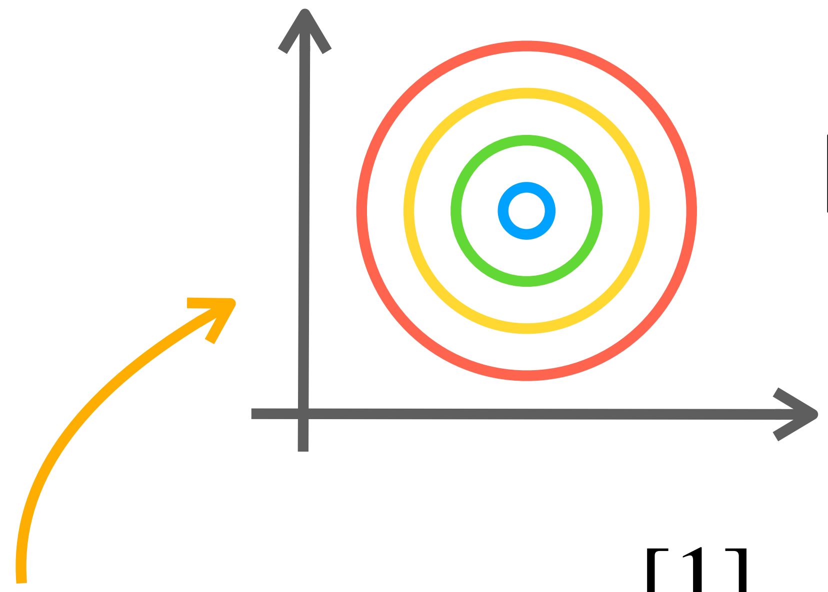
```
Parameter:
-----------
- X: dataset of size (d, m)

mean = np.mean(X, axis=1, keepdims=True)
std = np.std(X, axis=1, keepdims=True)
normalized = (X - mean) / (std + 1e-8)
```

Example 2: Normalizing Images

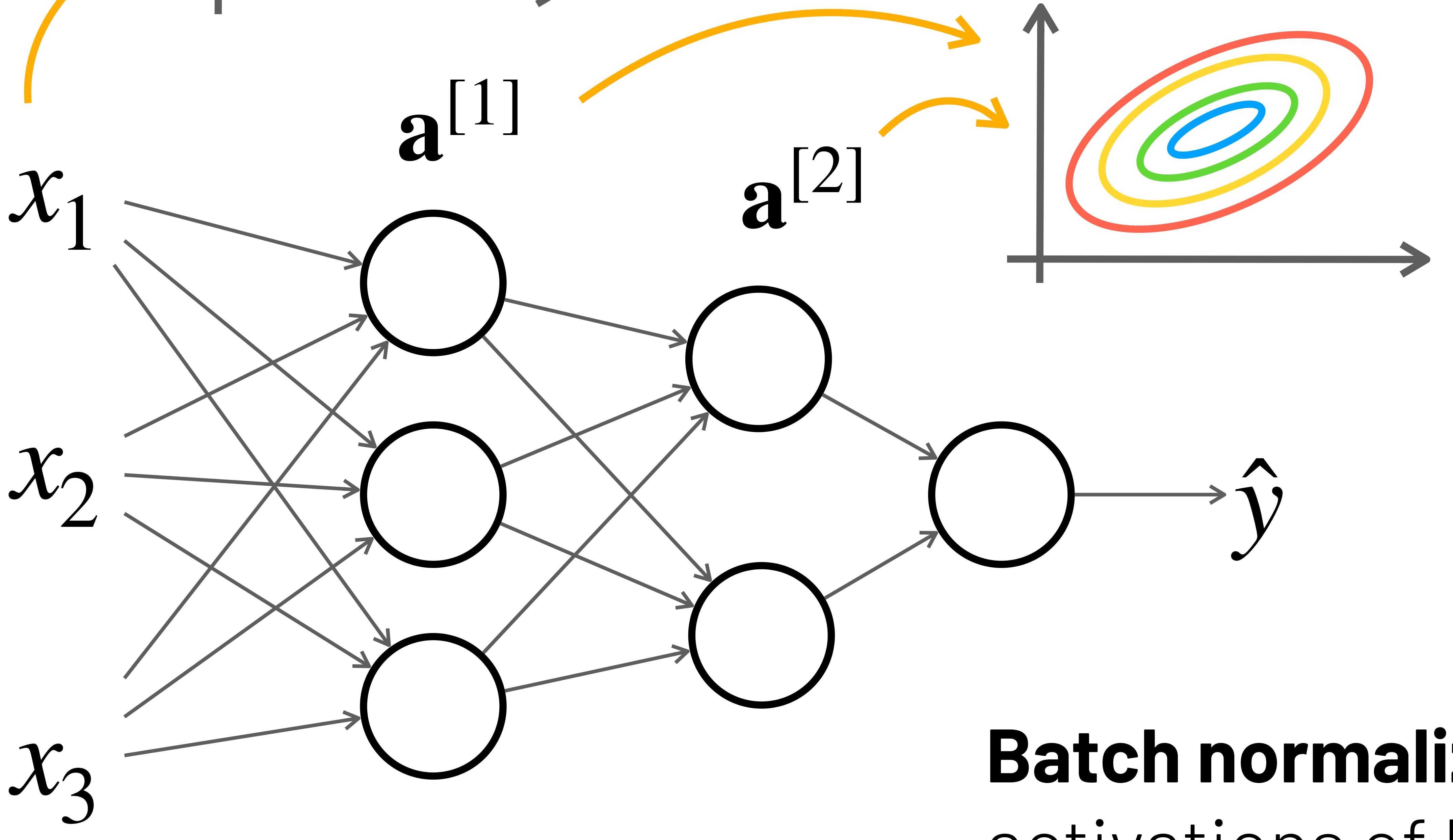
We can also apply this idea to normalize images, which can be done across channels or not:

```
Parameter:
-----------
- images : numpy.ndarray of shape (n_images, height, width, 3)

if normalization_type == 'zero_mean':
    # Zero mean and unit variance across all pixels and channels
    mean = np.mean(images)
    std = np.std(images)
    normalized = (images - mean) / (std + 1e-8)

elif normalization_type == 'zero_mean_per_channel':
    # Zero mean and unit variance per RGB channel
    mean = np.mean(images, axis=(0, 1, 2), keepdims=True)
    std = np.std(images, axis=(0, 1, 2), keepdims=True)
    normalized = (images - mean) / (std + 1e-8)
```

Batch Normalization



Input normalization helps only the first hidden layer..



...but it does not guarantee that the next activations will be normalized

Batch normalization is a technique to normalize the activations of hidden layers in neural networks

Ioffe, S. ; Szegedy, C. (2015). Batch Normalization: Accelerating Deep Network Training by Reducing Internal Covariate Shift. Proceedings of the 32nd International Conference on Machine Learning.

Batch Normalization

Given the linear outputs $\mathbf{z}^{[l](1)}, \mathbf{z}^{[l](2)}, \ldots, \mathbf{z}^{[l](m)}$ of a layer l for a minibatch with m examples, batch normalization normalizes $\mathbf{z}^{[l](i)}$ these values as follows:

Batch mean

$$\mu = \frac{1}{m} \sum_{i=1}^{m} \mathbf{z}^{[l](i)}$$

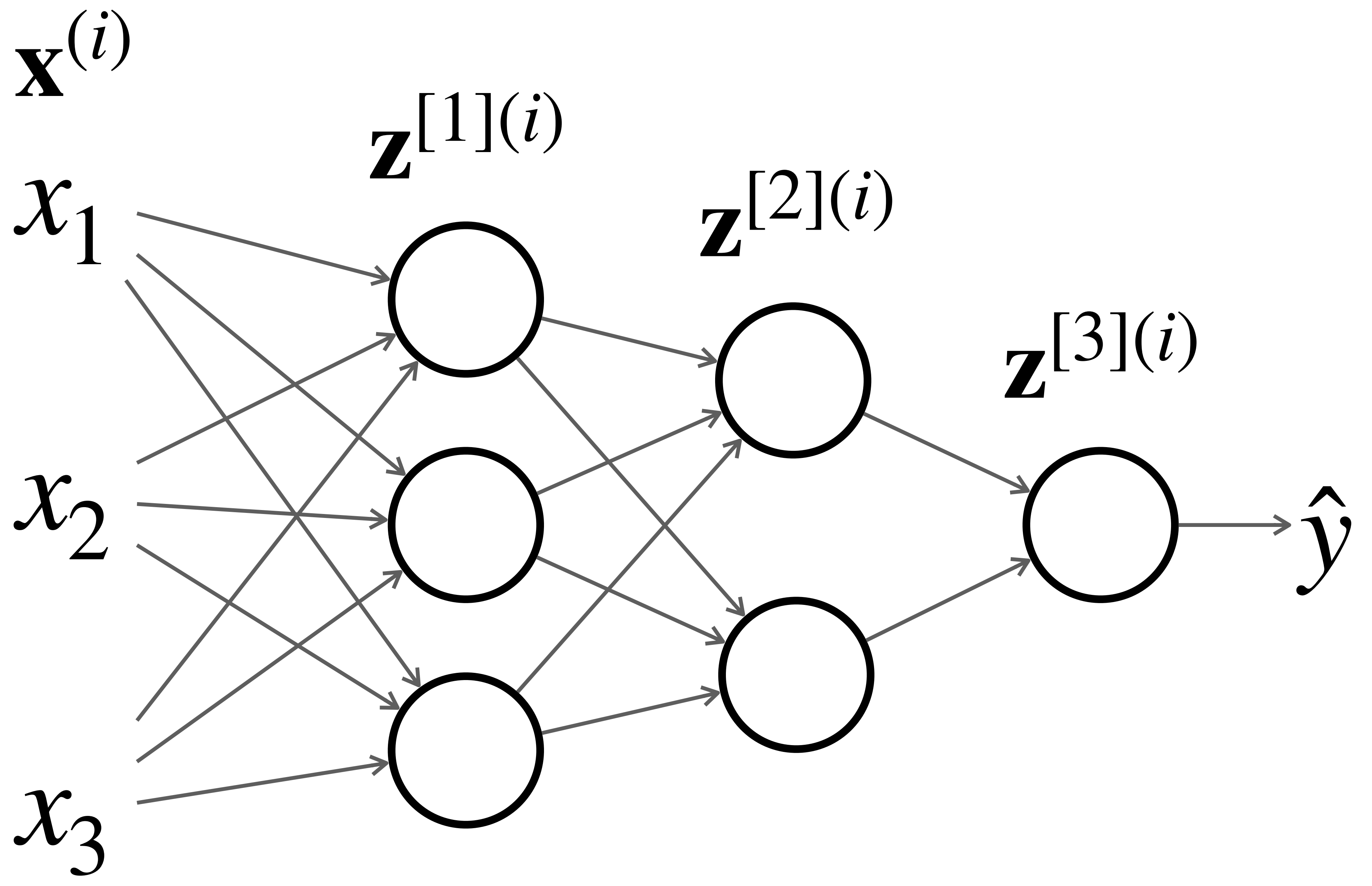
Batch variance

$$\sigma^2 = \frac{1}{m} \sum_{i=1}^{m} (\mathbf{z}^{[l](i)} - \mu)^2$$

$$\mathbf{z}^{[l](i)} = \frac{\mathbf{z}^{[l](i)} - \mu}{\sqrt{(\sigma^2) + \epsilon}}$$

Learnable parameters!

$$\tilde{\mathbf{z}}^{[l](i)} = \gamma \odot \mathbf{z}^{[l](i)} + \beta$$



Batch norm learn the mean β and variance γ of the activations!

Ioffe, S. ; Szegedy, C. (2015). Batch Normalization: Accelerating Deep Network Training by Reducing Internal Covariate Shift. Proceedings of the 32nd International Conference on Machine Learning.

Example: Batch Normalization

X

1.0	2.0	-1.0	0.0
0.5	1.0	0.5	-1.0
0.0	0.0	1.0	-0.5
$\mathbf{x}^{(1)}$	$\mathbf{x}^{(2)}$	$\mathbf{x}^{(3)}$	$\mathbf{x}^{(4)}$

$W^{[1]}$

0.1	0.2	-0.1
-0.2	0.1	0.2
0.1	-0.1	0.1

$b^{[1]}$

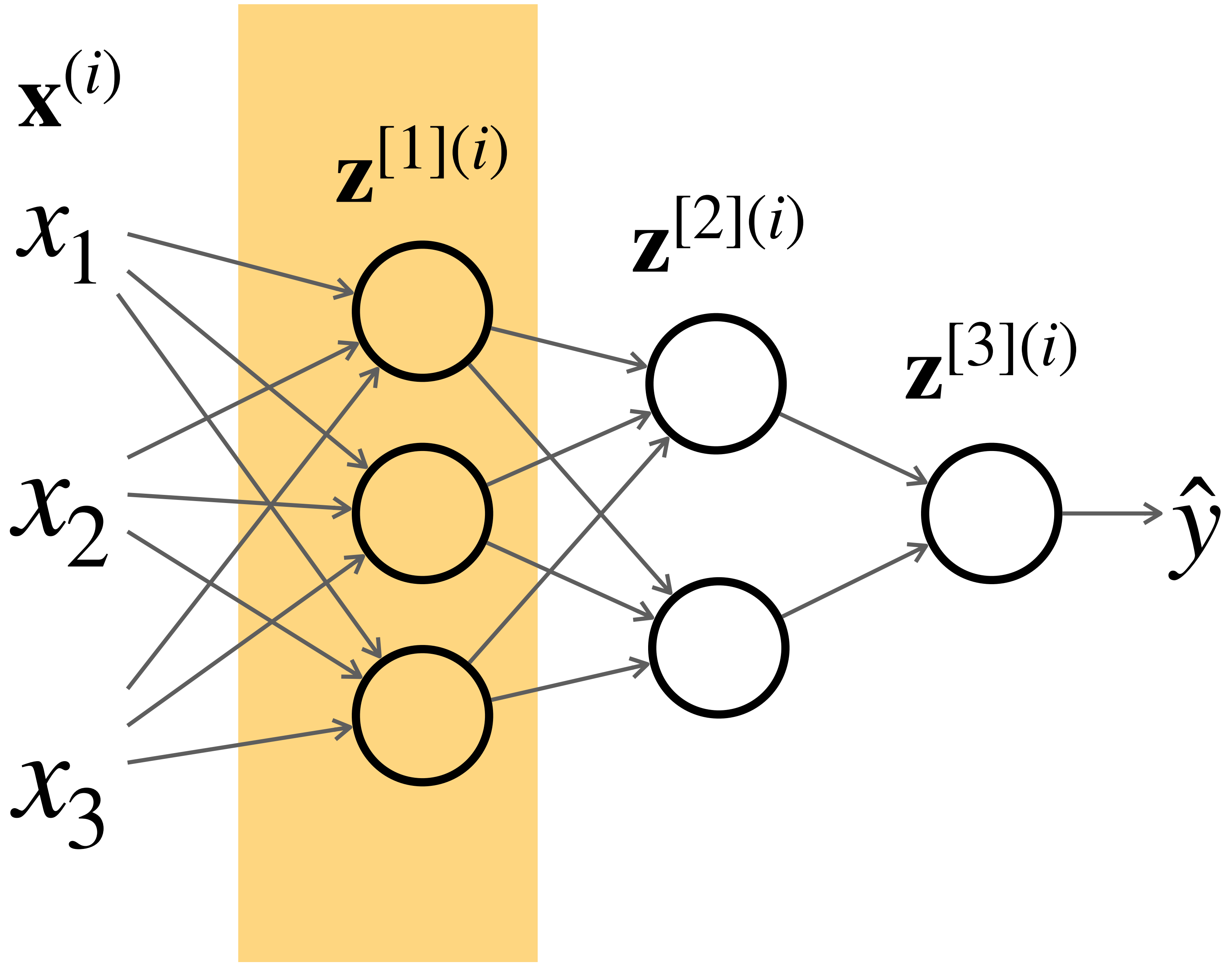
0
0
0

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

0.2	0.4	-0.1	-0.15
-0.15	-0.3	0.45	-0.2
0.05	0.1	-0.05	0.05
$\mathbf{z}^{[1](1)}$	$\mathbf{z}^{[1](2)}$	$\mathbf{z}^{[1](3)}$	$\mathbf{z}^{[1](4)}$

$$\mu = \frac{1}{m}\sum_{i=1}^{m}\mathbf{z}^{[l](i)}$$ **Batch mean**

$$= \frac{1}{4}\cdot\left(\begin{bmatrix}0.2\\-0.15\\0.05\end{bmatrix} + \begin{bmatrix}0.4\\-0.3\\0.1\end{bmatrix} + \begin{bmatrix}-0.1\\0.45\\-0.5\end{bmatrix} + \begin{bmatrix}-0.15\\-0.2\\0.05\end{bmatrix}\right) = \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}$$



Example: Batch Normalization

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

X

$\mathbf{x}^{(1)}$	$\mathbf{x}^{(2)}$	$\mathbf{x}^{(3)}$	$\mathbf{x}^{(4)}$
1.0	2.0	-1.0	0.0
0.5	1.0	0.5	-1.0
0.0	0.0	1.0	-0.5

$W^{[1]}$

0.1	0.2	-0.1
-0.2	0.1	0.2
0.1	-0.1	0.1

$b^{[1]}$

0
0
0

μ

0.08
-0.05
0.03

$Z^{[1]}$

$\mathbf{z}^{[1](1)}$	$\mathbf{z}^{[1](2)}$	$\mathbf{z}^{[1](3)}$	$\mathbf{z}^{[1](4)}$
0.2	0.4	-0.1	-0.15
-0.15	-0.3	0.45	-0.2
0.05	0.1	-0.05	0.05

$$\sigma^2 = \frac{1}{m}\sum_{i=1}^{m}(\mathbf{z}^{[l](i)} - \mu)^2$$ **Batch variance**

$$= \frac{1}{4}\cdot\left(\left(\begin{bmatrix}0.2\\-0.15\\0.05\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}\right)^2 + \left(\begin{bmatrix}0.4\\-0.3\\0.1\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}\right)^2 + \left(\begin{bmatrix}-0.1\\0.45\\-0.5\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}\right)^2 + \left(\begin{bmatrix}-0.15\\-0.2\\0.05\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}\right)^2\right) = \begin{bmatrix}0.05\\0.08\\0.02\end{bmatrix}$$

Example: Batch Normalization

Batch normalization

$$\mathbf{z}^{[l](i)} = \frac{\mathbf{z}^{[l](i)} - \mu}{\sqrt{(\sigma^2) + \epsilon}}$$

μ	σ^2
0.08	0.05
-0.05	0.08
0.03	0.02

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

$\mathbf{z}^{[1](1)}$	$\mathbf{z}^{[1](2)}$	$\mathbf{z}^{[1](3)}$	$\mathbf{z}^{[1](4)}$
0.2	0.4	-0.1	-0.15
-0.15	-0.3	0.45	-0.2
0.05	0.1	-0.05	0.05

$$\frac{\begin{bmatrix}0.2\\-0.15\\0.05\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}}{\sqrt{\begin{bmatrix}0.05\\0.08\\0.02\end{bmatrix}}} \quad \frac{\begin{bmatrix}0.4\\-0.3\\0.1\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}}{\sqrt{\begin{bmatrix}0.05\\0.08\\0.02\end{bmatrix}}} \quad \frac{\begin{bmatrix}-0.1\\0.45\\-0.05\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}}{\sqrt{\begin{bmatrix}0.05\\0.08\\0.02\end{bmatrix}}} \quad \frac{\begin{bmatrix}-0.15\\-0.2\\0.05\end{bmatrix} - \begin{bmatrix}0.08\\-0.05\\0.03\end{bmatrix}}{\sqrt{\begin{bmatrix}0.05\\0.08\\0.02\end{bmatrix}}} =$$

$$\gamma \qquad Z^{[1]}\text{ normalized} \qquad \beta$$

$$\begin{bmatrix}a1\\a2\\a3\end{bmatrix} \odot \begin{bmatrix}0.50 & 1.39 & -0.83 & -1.05\\-0.34 & -0.85 & 1.70 & -0.51\\0.22 & 1.14 & -1.60 & 0.22\end{bmatrix} + \begin{bmatrix}b1\\b2\\b3\end{bmatrix}$$

$\mathbf{z}^{[1](1)}$ $\mathbf{z}^{[1](2)}$ $\mathbf{z}^{[1](3)}$ $\mathbf{z}^{[1](4)}$

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Batch Normalization in Numpy

Batch normalization takes the mean and averages across the examples (axis = 1):

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

0.2	0.4	-0.1	-0.15
-0.15	-0.3	0.45	-0.2
0.05	0.1	-0.5	0.05
$\mathbf{z}^{[1](1)}$	$\mathbf{z}^{[1](2)}$	$\mathbf{z}^{[1](3)}$	$\mathbf{z}^{[1](4)}$

$Z^{[1]}$ normalized

0.50	1.39	-0.83	-1.05
-0.34	-0.85	1.70	-0.51
0.22	1.14	-1.60	0.22
$\mathbf{z}^{[1](1)}$	$\mathbf{z}^{[1](2)}$	$\mathbf{z}^{[1](3)}$	$\mathbf{z}^{[1](4)}$

```python
def batch_norm(Z, gamma, beta, epsilon=1e-8):
  m = Z.shape[1]

  # Calculate the mean
  mean = 1/m * np.sum(Z, axis=1, keepdims=True)

  # Calculate the variance
  variance = 1/m * np.sum((Z - mean)**2, axis=1, keepdims=True)

  # Normalize Z
  Z_norm = (Z - mean)/(np.sqrt(variance) + epsilon)

  # Rescale distribution to mean beta and variance gamma
  return gamma * Z_norm + beta
```

Batch Normalization in PyTorch

Defining a fully connected network in PyTorch with Batch Normalization:

```
# Define your neural network architecture with batch normalization
class MLP(nn.Module):
    def __init__(self):
        super().__init__()
        self.layers = nn.Sequential(
            nn.Flatten(),                # Flatten the input image tensor
            nn.Linear(28 * 28, 64),      # Fully connected layer from 28*28 to 64 neurons
            nn.BatchNorm1d(64),          # Batch normalization
            nn.ReLU(),                   # ReLU activation function
            nn.Linear(64, 32),           # Fully connected layer from 64 to 32 neurons
            nn.BatchNorm1d(32),          # Batch normalization
            nn.ReLU(),                   # ReLU activation function
            nn.Linear(32, 10)            # Fully connected layer from 32 to 10 neurons
        )

    def forward(self, x):
        return self.layers(x)
```

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Layer Normalization

Instead of normalizing across examples withing a mini-batch, layer normalization normalizes the activations across features, for each example separately:

Layer mean

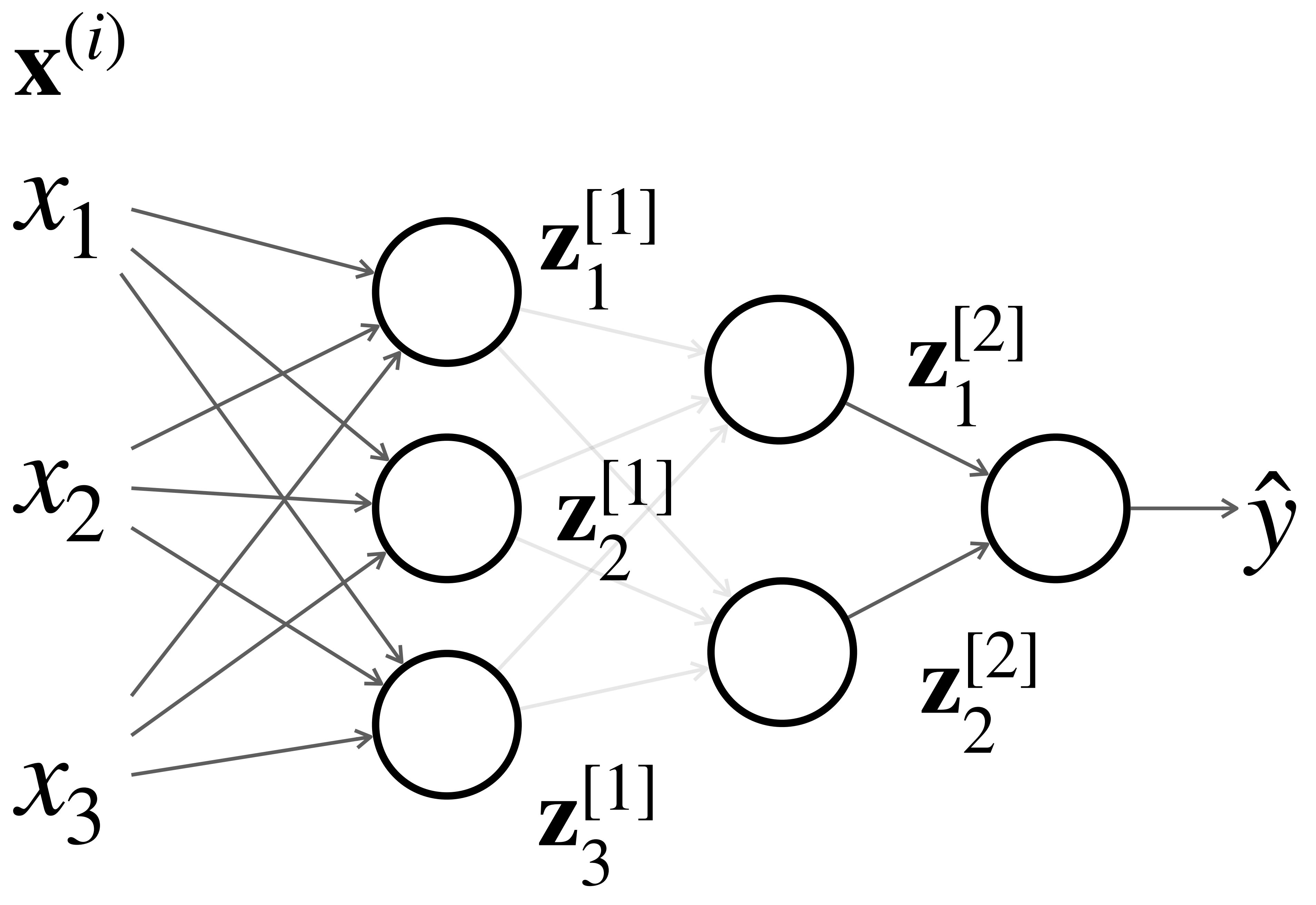
$$\mu = \frac{1}{n^{[l]}} \sum_{i=1}^{n^{[l]}} \mathbf{z}_i^{[l]}$$

Layer variance

$$\sigma^2 = \frac{1}{n^{[l]}} \sum_{i=1}^{n^{[l]}} (\mathbf{z}_i^{[l]} - \mu)^2$$

$$\mathbf{z}_i^{[l]} = \frac{\mathbf{z}_i^{[l]} - \mu}{\sqrt{(\sigma^2) + \epsilon}}$$

$$\tilde{\mathbf{z}}_i^{[l]} = \gamma \odot \mathbf{z}_i^{[l]\{i\}} + \beta$$



Why not batch norm? If the batch size is too small, then the estimates of mean and variance become too noisy

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Example: Layer Normalization

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

X

1.0	2.0	-1.0	0.0
0.5	1.0	0.5	-1.0
0.0	0.0	1.0	-0.5
$\mathbf{x}^{(1)}$	$\mathbf{x}^{(2)}$	$\mathbf{x}^{(3)}$	$\mathbf{x}^{(4)}$

$W^{[1]}$

0.1	0.2	-0.1
-0.2	0.1	0.2
0.1	-0.1	0.1

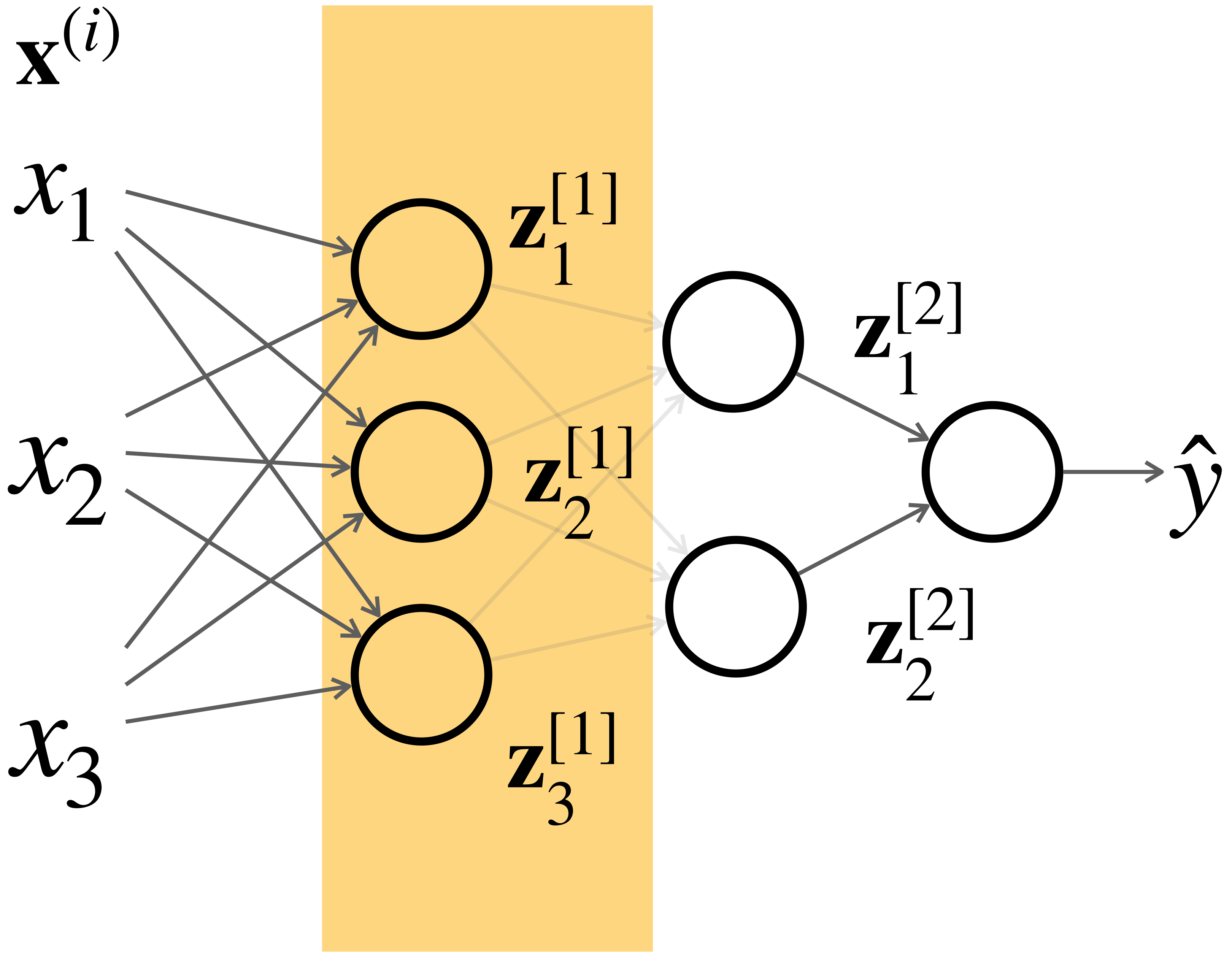
$b^{[1]}$

0
0
0

$\mathbf{z}_1^{[1]}$	0.2	0.4	-0.1	-0.15
$\mathbf{z}_2^{[1]}$	-0.15	-0.3	0.45	-0.2
$\mathbf{z}_3^{[1]}$	0.05	0.1	-0.05	0.05

$$\mu = \frac{1}{n^{[l]}} \sum_{i=1}^{n^{[l]}} \mathbf{z}_i^{[l]} = \frac{1}{3} \cdot \Big(\begin{bmatrix} 0.2 & 0.4 & -0.1 & -0.15 \end{bmatrix} + \begin{bmatrix} -0.15 & -0.3 & 0.45 & -0.2 \end{bmatrix} + \begin{bmatrix} 0.05 & 0.1 & -0.5 & 0.05 \end{bmatrix} \Big) = \begin{bmatrix} 0.03 & 0.06 & 0.1 & -0.1 \end{bmatrix}$$

Layer mean



Example: Layer Normalization

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

X

$\mathbf{x}^{(1)}$	$\mathbf{x}^{(2)}$	$\mathbf{x}^{(3)}$	$\mathbf{x}^{(4)}$
1.0	2.0	-1.0	0.0
0.5	1.0	0.5	-1.0
0.0	0.0	1.0	-0.5

$W^{[1]}$

0.1	0.2	-0.1
-0.2	0.1	0.2
0.1	-0.1	0.1

$b^{[1]}$

0
0
0

$Z^{[1]}$

$\mathbf{z}_1^{[1]}$	0.2	0.4	-0.1	-0.15
$\mathbf{z}_2^{[1]}$	-0.15	-0.3	0.45	-0.2
$\mathbf{z}_3^{[1]}$	0.05	0.1	-0.05	0.05

$$\sigma^2 = \frac{1}{n^{[l]}} \sum_{i=1}^{n^{[l]}} (\mathbf{z}_i^{[l]} - \mu)^2 = \frac{1}{3} \cdot \Big((\mathbf{z}_1^{[1]} - \mu)^2 + (\mathbf{z}_2^{[1]} - \mu)^2 + (\mathbf{z}_3^{[1]} - \mu)^2 \Big)$$

Layer variance

	$\mathbf{z}^{[1]}$					μ				
(	0.2	0.4	-0.1	-0.15	−	0.03	0.06	0.1	-0.1	)²
+ (	-0.15	-0.3	0.45	-0.2	−	0.03	0.06	0.1	-0.1	)²
+ (	0.05	0.1	-0.5	0.05	−	0.03	0.06	0.1	-0.1	)²

=

0.02	0.08	0.06	0.01

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Example: Layer Normalization

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

$\mathbf{z}_1^{[1]}$	0.2	0.4	-0.1	-0.15
$\mathbf{z}_2^{[1]}$	-0.15	-0.3	0.45	-0.2
$\mathbf{z}_3^{[1]}$	0.05	0.1	-0.05	0.05

Layer normalization

$$\mathbf{z}_i^{[l]} = \frac{\mathbf{z}_i^{[l]} - \mu}{\sqrt{(\sigma^2) + \epsilon}}$$

μ	0.03	0.06	0.1	-0.1
σ^2	0.02	0.08	0.06	0.01

$$\frac{[0.2 \;\; 0.4 \;\; -0.1 \;\; -0.15] - [0.03 \;\; 0.06 \;\; 0.1 \;\; -0.1]}{\sqrt{[0.02 \;\; 0.08 \;\; 0.06 \;\; 0.01]}}$$

$$\frac{[-0.15 \;\; -0.3 \;\; 0.45 \;\; -0.2] - [0.03 \;\; 0.06 \;\; 0.1 \;\; -0.1]}{\sqrt{[0.02 \;\; 0.08 \;\; 0.06 \;\; 0.01]}}$$

$$\frac{[0.05 \;\; 0.1 \;\; -0.5 \;\; 0.05] - [0.03 \;\; 0.06 \;\; 0.1 \;\; -0.1]}{\sqrt{[0.02 \;\; 0.08 \;\; 0.06 \;\; 0.01]}}$$

= $\gamma \odot Z^{[1]}$ normalized $+ \beta$

γ		$Z^{[1]}$ normalized				β
a1	$\mathbf{z}_1^{[1]}$	1.16	1.16	-0.80	-0.46	b1
a2	$\mathbf{z}_2^{[1]}$	-1.27	-1.27	1.40	-0.92	b2
a3	$\mathbf{z}_3^{[1]}$	0.11	0.11	-0.60	1.38	b3

Layer Normalization in Numpy

Layer normalization takes the mean and averages across the features (axis = 0):

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

$\mathbf{z}_1^{[1]}$	0.2	0.4	-0.1	-0.15
$\mathbf{z}_2^{[1]}$	-0.15	-0.3	0.45	-0.2
$\mathbf{z}_3^{[1]}$	0.05	0.1	-0.05	0.05

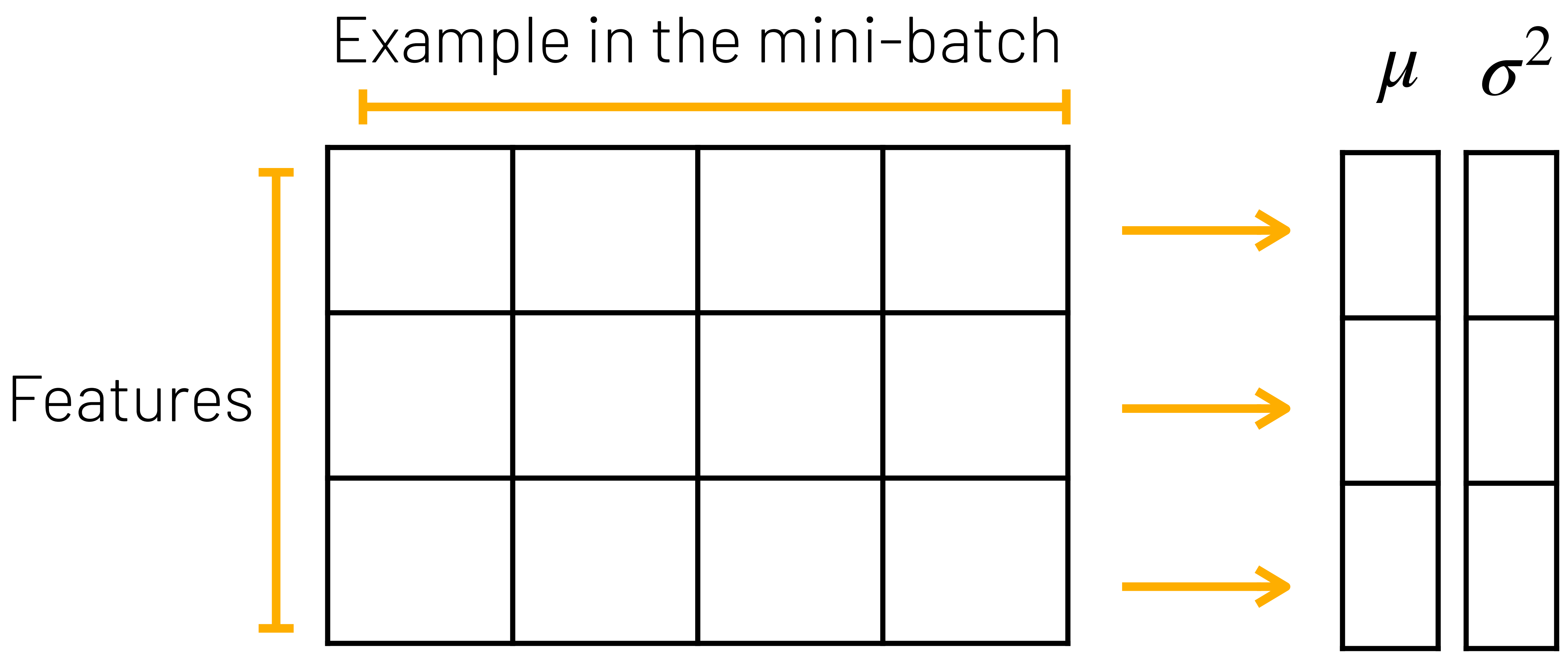
$Z^{[1]}$ normalized

$\mathbf{z}_1^{[1]}$	1.16	1.16	-0.80	-0.46
$\mathbf{z}_2^{[1]}$	-1.27	-1.27	1.40	-0.92
$\mathbf{z}_3^{[1]}$	0.11	0.11	-0.60	1.38

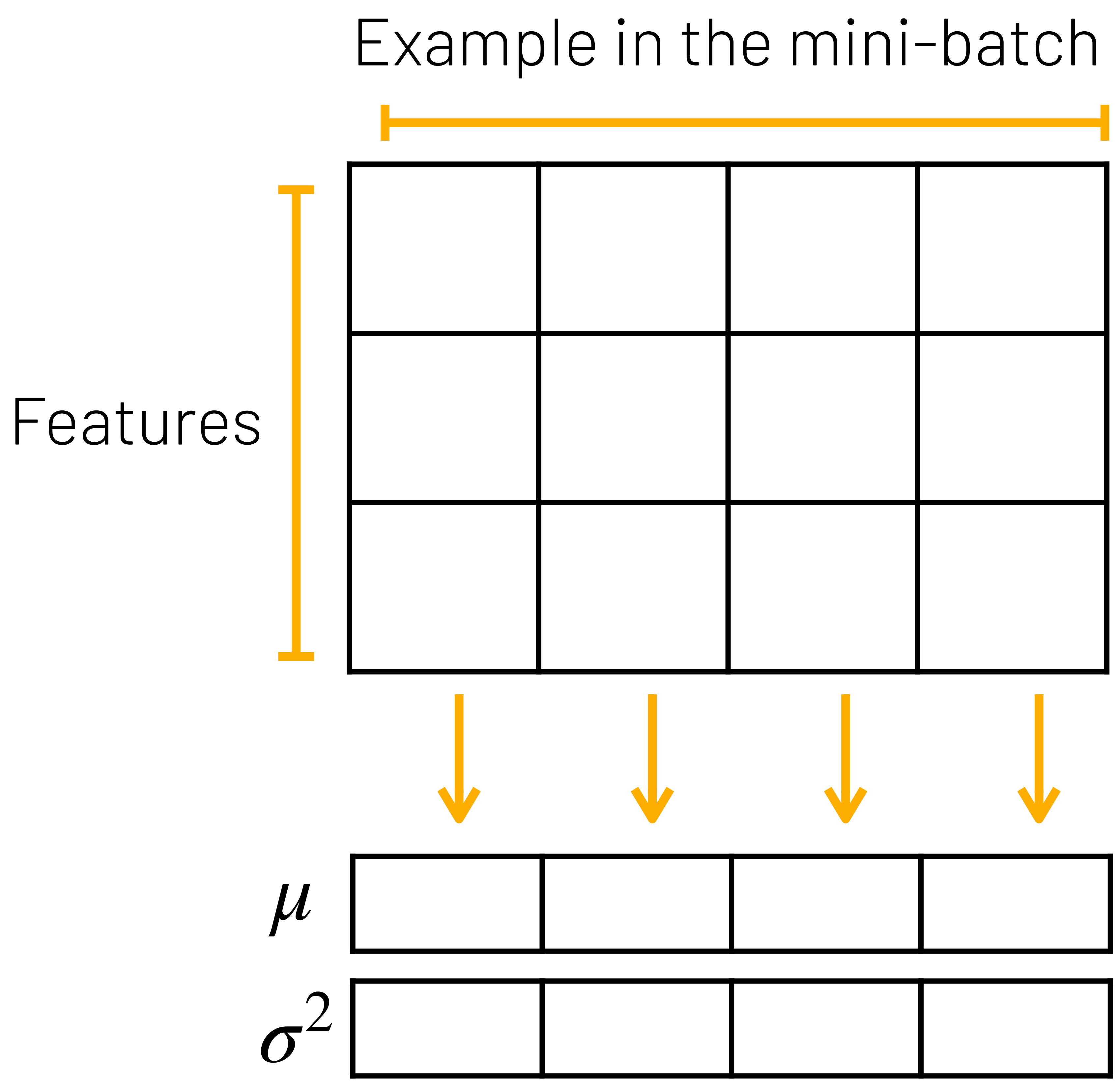
```python
def layer_norm(Z, gamma, beta, epsilon=1e-8):
  n = Z.shape[0]

  # Calculate the mean
  mean = 1/n * np.sum(Z, axis=0, keepdims=True)

  # Calculate the variance
  variance = 1/n * np.sum((Z - mean)**2, axis=0, keepdims=True)

  # Normalize Z
  Z_norm = (Z - mean)/(np.sqrt(variance) + epsilon)

  # Rescale distribution to mean beta and variance gamma
  return gamma * Z_norm + beta
```

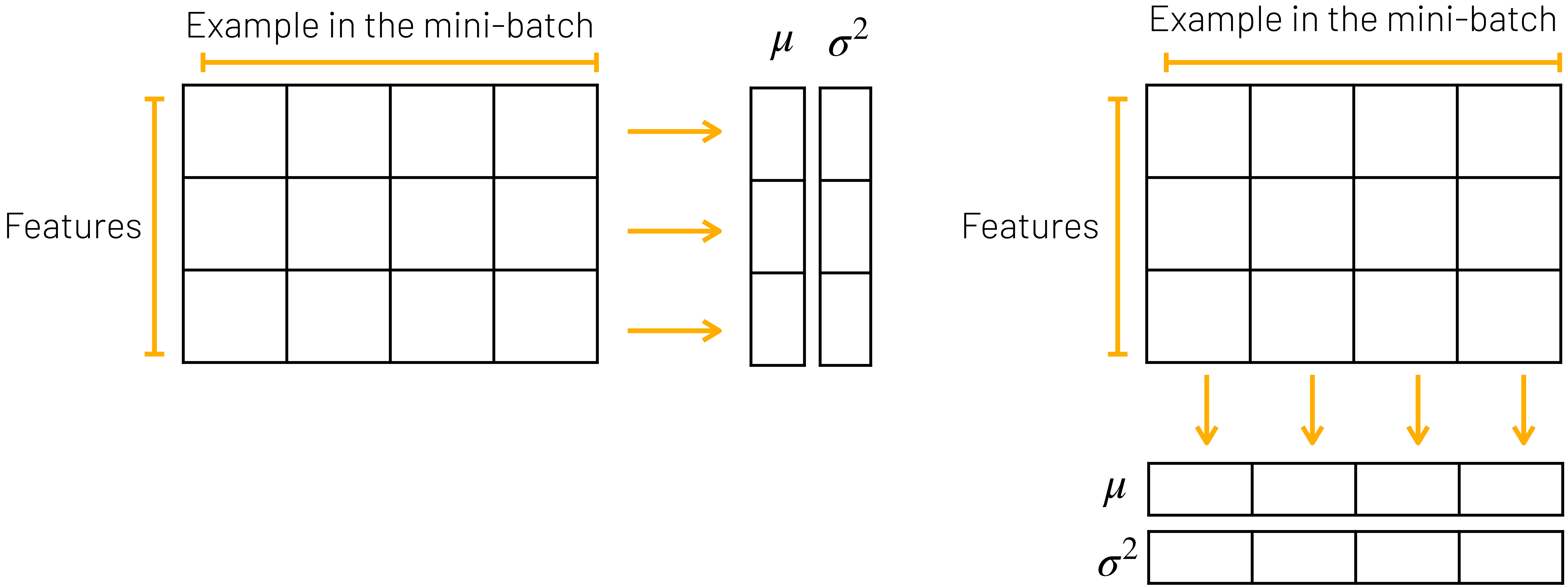
Batch Norm vs. Layer Norm

Batch norm: normalizes across the examples (axis = 1)



Layer norm: normalizes across the input features (axis = 0)





Example in the mini-batch
Features
μ
σ^2
Example in the mini-batch
Features
μ
σ^2

Next Lecture

L12: Recurrent Neural Networks

Sequential problems, basic recurrent neural networks, backpropagation through time, one-hot encodging, language models